Rost Multipliers of Lifted Kronecker Tensor Products

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Nonassociative Algebras and Geometry

Kronecker Tensor Product

$$\begin{aligned} \otimes\colon\ \mathrm{GL}_n \times \mathrm{GL}_m &\to \mathrm{GL}_{nm} \\ (A, B) &\mapsto A \otimes B \end{aligned}$$

$$A \otimes B = \left[\begin{array}{c:c:c} a_{11}B & \dots & a_{1n}B \\ \hdashline \vdots & & \vdots \\ \hdashline a_{n1}B & \dots & a_{nn}B \end{array}\right]$$

$$\otimes\colon\ \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \to \mathrm{SO}_{4nm}$$

$$\otimes\colon\ \mathrm{SO}_n \times \mathrm{SO}_m \to \mathrm{SO}_{nm}$$

Garibaldi's Example

$$\mathrm{PSp}_2 \times \mathrm{PSp}_8 \hookrightarrow \mathrm{HSpin}_{16} \to E_8$$

$Z(\mathrm{Spin}_{4n}) = \{1, -1, \xi, -\xi\} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
$\mathrm{SO}_{4n} = \mathrm{Spin}_{4n}/\{1, -1\}, \quad \mathrm{HSpin}_{4n} = \mathrm{Spin}_{4n}/\{1, \xi\}$

$$\begin{array}{ccc} & & \mathrm{Spin}_{16} \\ & \overset{\varphi}{\nearrow} & \downarrow \\ \mathrm{Sp}_2 \times \mathrm{Sp}_8 & \xrightarrow{\otimes} & \mathrm{SO}_{16} \end{array}$$

Garibaldi's Example

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$\mathrm{SO}_{4n} = \mathrm{Spin}_{4n} / \{1, -1\}, \quad \mathrm{HSpin}_{4n} = \mathrm{Spin}_{4n} / \{1, \xi\}$

$$\begin{array}{ccc} & \mathrm{Spin}_{16} & \\ \overset{\varphi}{\nearrow} & & \searrow\!\!\searrow \\ \mathrm{Sp}_2 \times \mathrm{Sp}_8 & \dashrightarrow & \mathrm{HSpin}_{16} \end{array}$$

$$\Rightarrow \mathrm{PSp}_2 \times \mathrm{PSp}_8 \hookrightarrow \mathrm{HSpin}_{16}$$

Chevalley Groups

Describes linear algebraic groups by generators and relations based off information from the root system.

Let G be a split semisimple linear algebraic group over $\mathbb{F} = \overline{\mathbb{F}}$, choose a maximal torus $T \subset G$. Then G has a root system $\Phi \subset T^*$.

Then we can express G as generated by symbols $x_\alpha(t)$ where $\alpha \in \Phi, t \in \mathbb{F}$.

$h_\alpha(t) := x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)x_\alpha(-1)x_{-\alpha}(1)x_\alpha(-1)$ for $t \in \mathbb{F}^\times$.

Chevalley Groups

Example: Sp_4. Type C_2, $\Phi = \{\pm e_1 \pm e_2, \pm 2e_1 \pm 2e_2\}$.

$$x_{e_1 - e_2}(t) = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_{-2e_2}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$h_{e_1 + e_2}(t) = \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{bmatrix}$$

Chevalley Groups

Theorem (Steinberg's Yale notes)

The simply connected groups of each type are given by:
$G = \langle x_\alpha(t) \mid \alpha \in \Phi, t \in \mathbb{F} \rangle$ *where*

- $x_\alpha(t)x_\alpha(u) = x_\alpha(t+u)$
- $h_\alpha(t)h_\alpha(u) = h_\alpha(tu)$
- $(x_\alpha(t), x_\beta(u)) = \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in\Phi}} x_{i\alpha+j\beta}(c_{ij}t^i u^j)$

After fixing an order on Φ, the constants c_{ij} only depend on the roots α, β and so can be calculated in non-simply connected groups of the same type.

Lifting $\otimes$

Let $\mathrm{Sp}_{2n}, \mathrm{Sp}_{2m}$ be generated by $x_\alpha(t), x'_\alpha(t)$, SO be generated by $y_\alpha(t)$, and Spin by $z_\alpha(t)$.

Describe $\otimes$: $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \to \mathrm{SO}_{4nm}$ in terms of Chevalley generators.
Example:

$$x_{e_i - e_j}(t) \otimes 1 = \prod_{k=1}^{2m} y_{e_{(i-1)2m+k} - e_{(j-1)2m+k}}(t)$$

$$1 \otimes x'_{2e_i}(u) = \prod_{k=0}^{n-1} y_{e_{2mk+i} - e_{2mk+m+i}}(-u)$$

Lifting $\otimes$

$$
\begin{array}{ccc}
 & & \mathrm{Spin}_{4nm} \\
 & \overset{\varphi}{\nearrow} & \Big\downarrow{\scriptstyle z_\alpha(t) \mapsto y_\alpha(t)} \\
\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} & \xrightarrow{\ \otimes\ } & \mathrm{SO}_{4nm}
\end{array}
$$

Naively define $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \to \mathrm{Spin}_{4nm}$ to act analogously on generators. Example:

$$x_{e_i - e_j}(t) \otimes 1 = \prod_{k=1}^{2m} y_{e_{(i-1)2m+k} - e_{(j-1)2m+k}}(t)$$

$$1 \otimes x'_{2e_i}(u) = \prod_{k=0}^{n-1} y_{e_{2mk+i} - e_{2mk+m+i}}(-u)$$

Lifting $\otimes$

$$\begin{array}{ccc} & & \mathrm{Spin}_{4nm} \\ & \overset{\varphi}{\nearrow} & \Big\downarrow {\scriptstyle z_\alpha(t) \mapsto y_\alpha(t)} \\ \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} & \xrightarrow{\otimes} & \mathrm{SO}_{4nm} \end{array}$$

Naively define $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \to \mathrm{Spin}_{4nm}$ to act analogously on generators. Example:

$$(x_{e_i - e_j}(t), 1) \mapsto \prod_{k=1}^{2m} z_{e_{(i-1)2m+k} - e_{(j-1)2m+k}}(t)$$

$$(1, x'_{2e_i}(u)) \mapsto \prod_{k=0}^{n-1} z_{e_{2mk+i} - e_{2mk+m+i}}(-u)$$

This does produce a well-defined homomorphism.

Lifting $\otimes$

Via the same process we can also lift $\otimes\colon \mathrm{SO}_n \times \mathrm{SO}_m \to \mathrm{SO}_{nm}$:

$$\begin{array}{ccc} \mathrm{Spin}_n \times \mathrm{Spin}_m & \overset{\varphi'}{\dashrightarrow} & \mathrm{Spin}_{nm} \\ \downarrow & & \downarrow \\ \mathrm{SO}_n \times \mathrm{SO}_m & \xrightarrow{\otimes} & \mathrm{SO}_{nm} \end{array}$$

Injections into HSpin

By composing φ, φ' with $\mathrm{Spin} \twoheadrightarrow \mathrm{HSpin}$ we can produce

- $\mathrm{PSp}_{2n} \times \mathrm{PSp}_{2m} \hookrightarrow \mathrm{HSpin}_{4nm}$ when n and/or m even.
- $\mathrm{PSO}_{2n} \times \mathrm{PSO}_{2m} \hookrightarrow \mathrm{HSpin}_{4nm}$ when n and/or m even.
- $\mathrm{HSpin}_{4n} \times \mathrm{SO}_{2m+1} \hookrightarrow \mathrm{HSpin}_{4n(2m+1)}$

Rost Multipliers

Let G be a split semisimple linear algebraic group. Associated to G is a group of quadratic invariants $Q(G)$ discussed by Garibaldi, Merkurjev, and Serre.

Let $T \subset G$ be the maximal torus, $\Phi \subset T^*$ the root system, and W the Weyl group.

W acts on Φ and this action can be extended to T^*, and from there extended to $\mathrm{Sym}(T^*)$.

$$Q(G) = \mathrm{Sym}^2(T^*)^W$$

Example: Killing form, $\sum_{\alpha \in \Phi} \alpha \otimes \alpha$.

Rost Multipliers

If G is simple then $Q(G) = \mathbb{Z}q$.
If $G = G_1 \times \ldots \times G_n$ is semisimple then $Q(G) = \mathbb{Z}q_1 \oplus \ldots \oplus \mathbb{Z}q_n \cong \mathbb{Z}^n$.

If $\rho\colon G \to H$ is a homomorphism, there is an associated homomorphism $\rho^*\colon Q(H) \to Q(G)$. The integers describing this map are called *Rost multipliers.*

$Q(G)$ is closely linked to the theory of degree 3 cohomological invariants of linear algebraic groups.

Multipliers of Injections into HSpin

$$\mathrm{PSp}_{2n} \times \mathrm{PSp}_{2m} \hookrightarrow \mathrm{HSpin}_{4nm}, \quad \mathrm{PSO}_{2n} \times \mathrm{PSO}_{2m} \hookrightarrow \mathrm{HSpin}_{4nm}$$

$$Q(\mathrm{HSpin}_{4nm}) \to Q(\mathrm{PSp}_{2n}) \oplus Q(\mathrm{PSp}_{2m})$$
$$Q(\mathrm{HSpin}_{4nm}) \to Q(\mathrm{PSO}_{2n}) \oplus Q(\mathrm{PSO}_{2m})$$
$$q \mapsto aq_1 \oplus bq_2$$

where $(a, b) =$ {

$n \pmod 4$	$m \pmod 4$: 0	1	2	3
0	(m, n)	$(m, \frac{n}{4})$	$(m, \frac{n}{2})$	$(m, \frac{n}{4})$
1	$(\frac{m}{4}, n)$		$(\frac{m}{2}, n)$	
2	$(\frac{m}{2}, n)$	$(m, \frac{n}{2})$	$(\frac{m}{2}, \frac{n}{2})$	$(m, \frac{n}{2})$
3	$(\frac{m}{4}, n)$		$(\frac{m}{2}, n)$	

Multipliers of Injections into HSpin

$$\mathrm{HSpin}_{4n} \times \mathrm{SO}_{2m+1} \hookrightarrow \mathrm{HSpin}_{4n(2m+1)}$$

$$Q(\mathrm{HSpin}_{4n(2m+1)}) \to Q(\mathrm{HSpin}_{4n}) \oplus Q(\mathrm{SO}_{2m+1})$$

$$q \mapsto \begin{cases} (2m+1)q_1 + 2nq_2 & n \equiv 0 \pmod 4 \\ (2m+1)q_1 + 4nq_2 & n \equiv 2 \pmod 4 \\ (2m+1)q_1 + 8nq_2 & n \equiv 1, 3 \pmod 4 \end{cases}$$

Cohomological Invariants

A degree 3 cohomological invariant of a linear algebraic group G with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$ is a natural transformation of functors

$$\alpha\colon H^1(-, G) \to H^3(-, \mathbb{Q}/\mathbb{Z}(2))$$

These form a group denoted $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))$.

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = \frac{\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}}{\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}}}$$

Cohomological Invariants

Merkurjev:

$$\mathrm{Inv}^3(\mathrm{PSp}_{2n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \equiv 0 \pmod 4 \\ 0 & \text{else} \end{cases}$$

$$\mathrm{Inv}^3(\mathrm{PSO}_{2n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \equiv 0 \pmod 4 \\ 0 & \text{else} \end{cases}$$

Bermudez and Ruozzi:

$$\mathrm{Inv}^3(\mathrm{HSpin}_{4n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \cong \begin{cases} 0 & n > 1 \text{ is odd or } n = 2 \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 2 \pmod 4, n \neq 2 \\ \mathbb{Z}/4\mathbb{Z} & n \equiv 0 \pmod 4 \end{cases}$$

Cohomological Invariants

A map of linear algebraic groups $\rho\colon G \to H$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} & \xrightarrow{\rho^*_{\mathrm{norm}}} & \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \\ \downarrow & & \downarrow \\ \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} & \xrightarrow{\rho^*_{\mathrm{ind}}} & \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \end{array}$$

In the case when G and H are split, ρ^*_{ind} is described by Rost multipliers.

A result of Merkurjev, Neshitov, and Zainoulline

They consider $\Delta \in \mathrm{Inv}^3(\mathrm{SL}_{n_1 n_2}/\mu_m, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ and $\otimes\colon \mathrm{SL}_{n_1}/\mu_m \times \mathrm{SL}_{n_2}/\mu_m \to \mathrm{SL}_{n_1 n_2}/\mu_m$

$$\begin{array}{ccccc}
 & & \xrightarrow{\otimes^*_{\mathrm{norm}}(\Delta)(\mathbb{F})} & & \\
H^1(\mathbb{F}, \frac{\mathrm{SL}_{n_1}}{\mu_m} \times \frac{\mathrm{SL}_{n_2}}{\mu_m}) & \longrightarrow & H^1(\mathbb{F}, \mathrm{SL}_{n_1 n_2}/\mu_m) & \xrightarrow{\Delta(\mathbb{F})} & H^3(\mathbb{F}, \mathbb{Q}/\mathbb{Z}(2)) \\
\downarrow & & \downarrow & \nearrow & \\
H^1(\mathbb{F}, \mathrm{PGL}_{n_1} \times \mathrm{PGL}_{n_2}) & \longrightarrow & H^1(\mathbb{F}, \mathrm{PGL}_{n_1 n_2}) & &
\end{array}$$

For a central simple $\mathbb{F}$-algebra A, define an element

$$\Delta(A) \in \frac{H^3(\mathbb{F}, \mathbb{Q}/\mathbb{Z}(2))}{\mathbb{F}^\times \cup \frac{m}{k}[A]}$$

A result of Merkurjev, Neshitov, and Zainoulline

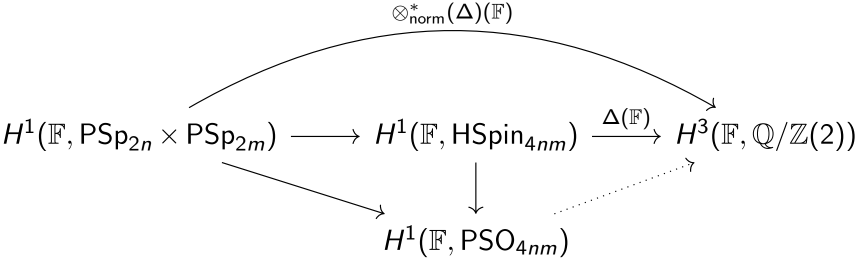
In this case, via Rost multipliers, $\otimes^*_{\mathsf{ind}} \equiv 0$

$$\Rightarrow \otimes^*_{\mathsf{norm}}(\Delta) \in \mathsf{Inv}^3(\frac{\mathsf{SL}_{n_1}}{\mu_m} \times \frac{\mathsf{SL}_{n_2}}{\mu_m}, \mathbb{Q}/\mathbb{Z}(2))_{\mathsf{dec}}$$

$$\Rightarrow \Delta(A_1 \otimes A_2) = \otimes^*_{\mathsf{norm}}(\Delta)(A_1, A_2) = 0 \in \frac{H^3(\mathbb{F}, \mathbb{Q}/\mathbb{Z}(2))}{\mathbb{F}^\times \cup \frac{m}{k}[A]}$$

Applications to HSpin

Consider $\Delta \in \mathrm{Inv}^3(\mathrm{HSpin}_{4nm}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ and
$\otimes\colon \mathrm{PSp}_{2n} \times \mathrm{PSp}_{2m} \hookrightarrow \mathrm{HSpin}_{4nm}$



For a central simple $\mathbb{F}$-algebra with orthogonal involution (A, σ), similarily define an element

$$\Delta(A) \in \frac{H^3(\mathbb{F}, \mathbb{Q}/\mathbb{Z}(2))}{\mathbb{F}^\times \cup d[A]}$$

Applications to HSpin

Consider $\Delta \in \mathrm{Inv}^3(\mathrm{HSpin}_{4nm}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ and
$\otimes\colon\ \mathrm{PSO}_{2n} \times \mathrm{PSO}_{2m} \hookrightarrow \mathrm{HSpin}_{4nm}$

$$\begin{tikzcd} H^1(\mathbb{F}, \mathrm{PSO}_{2n} \times \mathrm{PSO}_{2m}) \arrow[r] \arrow[rr, bend left, "\otimes^*_{\mathrm{norm}}(\Delta)(\mathbb{F})"] \arrow[dr] & H^1(\mathbb{F}, \mathrm{HSpin}_{4nm}) \arrow[r, "\Delta(\mathbb{F})"] \arrow[d] & H^3(\mathbb{F}, \mathbb{Q}/\mathbb{Z}(2)) \\ & H^1(\mathbb{F}, \mathrm{PSO}_{4nm}) \arrow[ur, dotted] & \end{tikzcd}$$

For a central simple $\mathbb{F}$-algebra with orthogonal involution (A, σ), similarily define an element

$$\Delta(A) \in \frac{H^3(\mathbb{F}, \mathbb{Q}/\mathbb{Z}(2))}{\mathbb{F}^\times \cup d[A]}$$

Thank You